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# Greeks and Partial Differential Equations for some Pricing Currency Options Models

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**Abstract**

In this study, we consider some pricing currency options models, which are using the Brownian motion, the fractional Broanian motion and the mixed fractional Brownian motion. The partial differential equations for values of European currency options and some Greeks are obtained for all these models. In addition, in the fractional environment, that parameter $H$ has huge effect on pricing options, the impact of the Hurst parameter $H$ is presented. Besides,comparing the Greeks for three currency options models are illustrated by some figures.

**Keywords:** Greeks, pricing options, currency options, fractional differential equations.

# 1. Introduction

In our research we use the following abbreviations and symbols

| | |
|---|---|
| $BS$ | Black-Scholes |
| $BM$ | Brownian motion |
| $GBM$ | Geometric Brownian motion |
| $FBM$ | Fractional Brownian motion |
| $MFBM$ | Mixed fractional Brownian motion |
| $JFBM$ | Jump fractional Brownian motion |
| $JMFBM$ | Jump mixed fractional Brownian motion |
| $B(t)$ | a standard Brownian motion |
| $B_H(t)$ | a fractional Brownian motion with parameter $H$ |
| $H \in (\frac{1}{2}, 1)$ | exponent parameter |
| $\mu$ | drift rate |
| $\sigma$ | volatility |
| $S_t$ | Spot price at time $t$ |
| $K$ | Strike price |
| $T$ | Expiration date |
| $r_d$ | Domestic interest rate |
| $r_f$ | Foreign interest rate |
| $C(t, S_t)$ | Value of European call currency option |
| $P(t, S_t)$ | Value of European put currency option |

| | |
|---|---|
| $V(t, S_t)$ | Value of a currency option that depends just $t$ and $S_t$ |
| $f(S_t)$ | Bounded payoff of a currency option |
| $\Phi(.)$ | Cumulative normal distribution |
| $N(t)$ | Poisson process with rate $\lambda \in (0, 1)$ |
| $J(t)$ | Jump size at time $t$ |
| $\varepsilon_n$ | Expectation operator over the distribution of $\prod_{i=1}^{n} e^{J(t_i)}$ |
| $\Delta$ | Sensitivity to the underlying price |
| $\Gamma$ | Sensitivity to the underlying price sensitivity |
| $\upsilon$ | Sensitivity to the volatility |
| $\Theta$ | Sensitivity to the time expiration |
| $\rho$ | Sensitivity to the interest rate |

A currency options refers an agreement that gives right to the holder in order to buy or sell a determined amount of foreign currency at a constant exercise price on option exercise. American options are traded at any time before they expire. European options can be exercised only during a specified period immediately before expiration.

Option pricing developed by Black and Scholes (1973) , is one of the most frequently used formulas in the area of financial mathematics. This method is based on the $GBM$ as follows

$$dS_t = \mu S_t dt + \sigma S_t dB(t), \qquad S_0 = S > 0, \quad 0 \leq t \leq T \tag{1}$$

here $\mu, \sigma$ are constant.

Nowadays, the $BS$ model is the most commonly used model for analyzing financial data. However, scientists have argued that option pricing, utilizing $BS$ model based on $BM$, is not able to assess some components of financial data, including:heavy tailed, long-range dependence, and etc. Hence, scholars have presented some generalizations of the $BS$ model in order to capture these phenomena observed on stock markets Dravid et al. (1993), Duan and Wei (1999), Garman and Kohlhagen (1983), Ho et al. (1995), Toft and Reiner (1997). Moreover, Gu et al. (2012) modified the $BS$ model to evaluate some components of financial assets such as long-range correlations and self similarity . They introduced the $FBM$ model by replacing $BM$ with $FBM$ in the standard $BS$ model as follows

$$dS_t = \mu S_t dt + \sigma S_t dB_H(t), \qquad S_0 = S > 0, \quad 0 \leq t \leq T. \tag{2}$$

Xiao et al. (2010) investigated the new model with combination of the jump process and $FBM$ model in order to get behavior from financial markets such as:discontinous or jumps, long memory and self-similarity. Unfortunately, owing to $FBM$ is neither a semi-martingale nor a Markov process, some studies Cheridito (2003), Rogers et al. (1997) substantiated that the $FBM$ model affirms arbitrage in a complete and frictionless market. As a result, the $MFBM$ model Mishura and Mishura (2008), Shokrollahi and Kılıçman (2014a) has been introduced to resolve this obstacle and to consider the long memory feature,and also to capture the fluctuations from stock markets.

The $MFBM$ is a linear composition of $BM$ and $FBM$, which displays long-range correlation and fluctuations from financial stock markets. The $MFBM$ model is defined as follows

$$dS_t = \mu S_t dt + \sigma S_t dB(t) + \sigma S_t dB_H(t), \qquad S_0 = S > 0, \quad 0 \leq t \leq T, \tag{3}$$

where $B(t)$ and $B_H(t)$ are assumed independent. Furthermore, Shokrollahi and Kılıçman (2014b, 2015) have introduced a new framework for pricing currency options by combining the $MFBM$ model and the Poisson jump process to get discontinuous, fluctuations, and the long memory feature from stock markets. In this study we consider various pricing currency option models which are used the $BS$ model, the $FBM$ model , $JFBM$ model , $MFBM$ model and $JMFBM$ model, then we investigate the partial differential equations that the value of currency options satisfy them. Furthermore, for all considered models the Greeks are obtained.

The remainder of this study as follow: Section 2 provides some definitions and some features of Greeks. In Section 3, some pricing currency options methods are presented. Moreover, the partial differential equation and Greeks for these methods and the impact of exponent parameter $H$ on pricing models are obtained. Furthermore, the differences among some Greeks are shown. Finally, Section 4 concludes the paper.

# 2. Preliminaries

Greeks summarize how option prices change with respect to underlying variables and are critically important in asset pricing and risk management. It can be used to rebalance the portfolio to achieve desired exposure to a certain risk. More importantly, knowing the Greek, a particular exposure can be hedged from adverse changes in the market by using appropriate amount of other related financial instruments. Unlike option prices, which can be observed in the market, Greeks can not be observed and have to be calculated given a model assumption. Typically, the Greeks are computed using a partial differentiation of the price formula.

**Delta**

Delta ($\Delta$) of an option defined as

$$\Delta = \frac{change\, in\, option\, price}{change\, in\, underlying} \tag{4}$$

The sensitivity of the option to the underlying finance is assessed by Delta. In this regard, call deltas indicates positive contrary to put deltas that is negative.It should be noted that there is a positive interaction between call option price and the underlying asset in comparison to inversely relation of put option price with the underlying asset. In view of this, according to the put-call-parity, we have the put delta equals the call delta minus 1. Delta Hedging is a very common strategy to do the arbitrage and minimizes the risk of portfolio in the option market.

**Gamma**

Gamma ($\Gamma$) calculated the the immediate changes of delta in terms of partial alterations, which occur in the underlying price of stock. It is the second derivative of the option value with respect to the underlying asset.

$$\Gamma = \frac{change\, in\, delta}{change\, in\, underlying} \tag{5}$$

It reveals that for saving the delta in a neural position, knowing how much and how often a position should be hedged repeatedly is essential. The delta-hedge strategy in order to hedge a portfolio, by keeping gamma in a small position, since the smaller is not common, so we intend to regulate the hedge to save the delta in a neural manner. The gammas are always positive for call options, while be negative for put options. However, gammas generally change signs for more complicated options such as binary options.

**Theta**

Theta ($\Theta$) is defined as

$$\Theta = -\frac{change\, in\, option\, price}{change\, in\, time\, to\, maturity} \tag{6}$$

Theta measures the sensitivity of the value of the option to the change of time to maturity. If the asset price is constant, consequently the option will change by theta with time.

**Vega**

The Vega, assesses the sensitivity to volatility, which expresses as the amount of money per stock gain or lose as volatility increases or decreases by 10/0. It is the derivative of the value of the option in terms of the volatility of the stock price . The positive value of Vega substantiates that the value of an option increase augmenting the volatility in comparison to the negative value of Vega, which indicates that the value of an option will reduce by declining of the volatility. Greater volatility reflects higher value of call and put options.

$$Vega(v) = \frac{change\, in\, option\, price}{change\, in\, volatility} \tag{7}$$

**Rho**

Rho ($\rho$) refers to the rate of option alteration respect to the rate of interest.

$$\rho = \frac{change\, in\, option\, price}{change\, in\, interest\, rate} \tag{8}$$

# 3. Greeks and Partial Differential Equations

In the following section we want to obtain the Greeks and partial differential equations for these models: the $BS$ model, the $FBM$ model, the $JFBM$ model , the $MFBM$ model and, the $JMFBM$ model. Moreover, the partial differential equation for any pricing currency models are presented. For all the pricing currency models the following hypothesis will be provided:

(i) there is no transaction costs or taxes;

(ii) security trading is continuous;

(iii) here $r_d$ and $r_f$ are known and constant thought time;

(iv) there are no risk-free arbitrage opportunities.

Now, we consider two investments:

1. a money market account:

$$dM_t = r_d M_t dt, \quad M_0 = 1, \quad 0 \leq t \leq T. \tag{9}$$

2. a stock whose price satisfies in the following pricing models.

## 3.1 The $BS$ model

The $BS$ model for the first time has been presented by Black and Scholes (1973). The price of stock in this model that follows the equation (1) is given by

$$dS_t = (r_d - r_f)S_t dt + \sigma dB_t \quad S_0 = 1, \quad 0 \leq t \leq T. \tag{10}$$

Then, the pricing of call currency option as follows

$$C(t, S_t) = S_t e^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \tag{11}$$

and for put currency options

$$P(t, S_t) = Ke^{-r_d(T-t)}\Phi(-d_2) - S_t e^{-r_f(T-t)}\Phi(-d_1), \tag{12}$$

here

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{K} + (r_d - r_f)(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{13}$$

**Theorem 3.1.** For the $BS$ model, $V(t, S_t)$ is the solution of the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r_d - r_f)\frac{\partial V}{\partial S} - r_d V = 0. \tag{14}$$

**Theorem 3.2.** The Greeks for the $BS$ call currency options model are given by

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial S_t} = e^{-r_f(T-t)}\Phi(d_1), \\ \nabla &= \frac{\partial C}{\partial K} = -e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_d} &= \frac{\partial C}{\partial r_d} = K(T-t)e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_f} &= \frac{\partial C}{\partial r_f} = S_t(T-t)e^{-r_f(T-t)}\Phi(d_1), \\ \Theta &= \frac{\partial C}{\partial t} = S_t r_f e^{-r_f(T-t)}\Phi(d_1) - K r_d e^{-r_d(T-t)}\Phi(d_2) \\ &- S_t e^{-r_f(T-t)}\frac{\sigma}{2\sqrt{T-t}}\Phi'(d_1), \\ \Gamma &= \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)}\frac{\Phi'(d_1)}{S_t\sqrt{\sigma^2(T-t)}}, \\ \vartheta_\sigma &= \frac{\partial C}{\partial \sigma} = S_t e^{-r_f(T-t)}\sqrt{T-t}\,\Phi'(d_1). \end{aligned}$$

## 3.2 The $FBM$ model

The $FBM$ model is a generalization of the $BS$ model, which is based on replacing the $BM$ by a $FBM$ in the $BS$ model Cheridito (2003), Dasgupta and Kallianpur (2000), Rogers et al. (1997), Salopek (1998), Shiryaev (1998). Necula (2002) has represented the $FBM$ for pricing options. The price of stock in the $FBM$ model based on equation () is given by

$$dS_t = (r_d - r_f)S_t dt + \sigma S_t dB_H(t), \qquad S_0 = S > 0, \quad 0 \le t \le T. \tag{15}$$

Then, the pricing of call currency option as follows

$$C(t, S_t) = S_t e^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \tag{16}$$

and for put currency options

$$P(t, S_t) = Ke^{-r_d(T-t)}\Phi(-d_2) - S_t e^{-r_f(T-t)}\Phi(-d_1), \tag{17}$$

here

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{K} + (r_d - r_f)(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}, \\ d_2 &= d_1 - \sigma\sqrt{T^{2H} - t^{2H}}. \end{aligned} \tag{18}$$

**Theorem 3.3.** For the $FBM$ model, $V(t, S_t)$ is the solution of the $PDE$

$$\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f)S_t \frac{\partial V}{\partial S_t} - r_d V = 0. \tag{19}$$

**Theorem 3.4.** The Greeks for the $FBM$ call currency option model can be written as

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial S_t} = e^{-r_f(T-t)}\Phi(d_1), \\ \nabla &= \frac{\partial C}{\partial K} = -e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_d} &= \frac{\partial C}{\partial r_d} = K(T-t)e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_f} &= \frac{\partial C}{\partial r_f} = S_t(T-t)e^{-r_f(T-t)}\Phi(d_1), \\ \Theta &= \frac{\partial C}{\partial t} = S_t r_f e^{-r_f(T-t)}\Phi(d_1) - Kr_d e^{-r_d(T-t)}\Phi(d_2) \\ &- S_t e^{-r_f(T-t)}\frac{\sigma H t^{2H-1}}{\sqrt{T^{2H} - t^{2H}}}\Phi'(d_1), \\ \Gamma &= \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)}\frac{\Phi'(d_1)}{S_t\sqrt{\sigma^2(T^{2H} - t^{2H})}}, \\ \vartheta_\sigma &= \frac{\partial C}{\partial \sigma} = S_t e^{-r_f(T-t)}\sqrt{T^{2H} - t^{2H}}\Phi'(d_1). \end{aligned}$$

The following theorem represents the influence of Hurst parameter $H$ in $FBM$ model.

**Theorem 3.5.** The impact of the parameter $H$ is given by

$$\frac{\partial C}{\partial H} = S_t e^{-r_f(T-t)} \frac{\sigma(T^{2H}\ln T - t^{2H}\ln t)}{\sqrt{T^{2H}-t^{2H}}} \Phi'(d_1). \tag{20}$$

## 3.3 The $JFBM$ model

For capture jumps and discontinuous from financial markets Xiao et al. (2010) introduced the new model by combination of the Poisson jump process and $FBM$ model. The value of stock in this model is

$$\begin{aligned} dS_t &= (r_d - r_f)S_t dt + \sigma S_t dB_H(t) \\ &+ (e^{J(t)} - 1)dN_t, \qquad S_0 = S > 0, \quad 0 \le t \le T, \end{aligned} \tag{21}$$

where $(e^{J(t)} - 1) \sim N(\mu_{J(t)}, \delta^2(t))$. Moreover, all three source of $B_H(t)$, $N(t)$, and $e^{J(t)} - 1$, are supposed to be independent.

Thus,the price of call currency options is given by

$$\begin{aligned} C(t, S_t) &= \sum_{n=0}^{\infty} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \varepsilon_n \\ &\times \Big[ S_t \prod_{i=1}^{n} e^{J(t_i)} e^{(-r_f + \lambda\mu_{J(t)})(T-t)} \Phi(d_1) \\ &- K e^{-r_d(T-t)} \Phi(d_2) \Big], \end{aligned} \tag{22}$$

and for the put currency option we have

$$\begin{aligned} P(t, S_t) &= \sum_{n=0}^{\infty} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \varepsilon_n \\ &\times \Big[ K e^{-r_d(T-t)} \Phi(-d_2) \\ &- S_t \prod_{i=1}^{n} e^{J(t_i)} e^{(-r_f + \lambda\mu_{J(t)})(T-t)} \Phi(-d_1) \Big], \end{aligned} \tag{23}$$

here

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_t \prod_{i=1}^{n} e^{J(t_i)}}{K} + (r_d - r_f - \lambda\mu_{J(t)})(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}, \\ d_2 &= d_1 - \sigma\sqrt{T^{2H} - t^{2H}}. \end{aligned} \tag{24}$$

**Theorem 3.6.** For the $JFBM$ model, $V(t, S_t)$ is the solution of the $PDE$

$$\begin{aligned} &\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S^2 \frac{\partial^2 V}{\partial S^2} + (r_d - r_f - \lambda\mu_{J(t)}) S \frac{\partial V}{\partial S} \\ &-r_d V + \lambda E[V(e^{J(t)} S_t, t) - V(S_t, t)] = 0. \end{aligned} \tag{25}$$

**Theorem 3.7.** The Greeks for the $JFBM$ call currency option model are given by

$$\begin{aligned} \triangle &= \frac{\partial C}{\partial S} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \\ &\left[\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f + \lambda\mu_{J(t)})(T-t)} \Phi(d_1)\right]. \\ \nabla &= \frac{\partial C}{\partial K} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \left[-e^{-r_d(T-t)}\Phi(d_2)\right]. \\ \rho_{r_d} &= \frac{\partial C}{\partial r_d} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \left[K(T-t)e^{-r_d(T-t)}\right]\Phi(d_2). \\ \rho_{r_f} &= \frac{\partial C}{\partial r_f} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \left[-S_t(T-t)\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f + \lambda\mu_{J(t)})(T-t)} \Phi(d_1)\right]. \end{aligned}$$

$$\begin{aligned} \Theta &= \frac{\partial C}{\partial t} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^{n+1}(T-t)^n - ne^{-\lambda(T-t)}\lambda^n (T-t)^{n-1}}{n!} \varepsilon_n \\ &\left[S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f + \lambda\mu_{J(t)})(T-t)} \Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2)\right] + \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \\ &\left[(r_f + \lambda\mu_{J(t)}) S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f + \lambda\mu_{J(t)})(T-t)} \Phi(d_1) - r_d K e^{-r_d(T-t)}\Phi(d_2)\right. \\ &\left. - \frac{H\sigma^2 t^{2H-1}}{\sqrt{\sigma^2(T^{2H} - t^{2H})}} S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f + \lambda\mu_{J(t)})(T-t)} \Phi'(d_1)\right]. \end{aligned}$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \left[ \frac{\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_J(t))(T-t)}}{S_t\sqrt{\sigma^2(T^{2H}-t^{2H})}} \Phi'(d_1) \right]$$

$$\nu_\sigma = \frac{\partial C}{\partial \sigma} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \left[ S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)} \sqrt{T^{2H}-t^{2H}} \Phi'(d_1) \right].$$

**Theorem 3.8.** The effect of hurst parameter as follows

$$\frac{\partial C}{\partial H} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \varepsilon_n \Big[ S_t \frac{\sigma(T^{2H}\ln T - t^{2H}\ln t)}{\sqrt{T^{2H}-t^{2H}}} \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)} \Phi'(d_1) \Big]. \quad (26)$$

## 3.4 The $MFBM$ model

The $MFBM$ model is a extension of the standard $BS$ model based on the $MFBM$ which is a linear combination of the $FBM$ and the $BM$. The first study of applying $MFBM$ model in finance is presented by Cheridito (2003). He showed that, for $H \in (3/4, 1)$, the $MFBM$ was equivalent to one with $BM$ and then it was arbitrage free. For this we assume that $H \in (3/4, 1)$, for get more information about the $MFBM$ you can see Cheridito (2003), Shokrollahi and Kılıçman (2014a,a), Sun (2013), Xiao et al. (2012), Zili (2006) .

The price of stock in the $MFBM$ model respect to the equation (3) can be written as follows

$$dS_t = (r_d - r_f)S_t dt + \sigma S_t dB(t) + \sigma S_t dB_H(t), \quad S_0 = S > 0, \quad 0 \le t \le T. \quad (27)$$

Hence, the pricing of call currency option is given by

$$C(t, S_t) = S_t e^{-r_f(T-t)}\Phi(d_1) - K e^{-r_d(T-t)}\Phi(d_2), \quad (28)$$

and for put currency options

$$P(t, S_t) = Ke^{-r_d(T-t)}\Phi(-d_2) - S_t e^{-r_f(T-t)}\Phi(-d_1), \tag{29}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{K} + (r_d - r_f)(T-t) + \frac{\sigma^2}{2}(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T - t + T^{2H} - t^{2H}}}, \\ d_2 &= d_1 - \sigma\sqrt{T - t + T^{2H} - t^{2H}}. \end{aligned} \tag{30}$$

**Theorem 3.9.** For the $MFBM$ model, $V(t, S_t)$ is the solution of the $PDE$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f) S_t \frac{\partial V}{\partial S_t} - r_d V = 0. \tag{31}$$

**Theorem 3.10.** The Greeks for the $MFBM$ call currency option model are given by

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial S_t} = e^{-r_f(T-t)}\Phi(d_1), \\ \nabla &= \frac{\partial C}{\partial K} = -e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_d} &= \frac{\partial C}{\partial r_d} = K(T-t)e^{-r_d(T-t)}\Phi(d_2), \\ \rho_{r_f} &= \frac{\partial C}{\partial r_f} = S_t(T-t)e^{-r_f(T-t)}\Phi(d_1), \\ \Theta &= \frac{\partial C}{\partial t} = S_t r_f e^{-r_f(T-t)}\Phi(d_1) - K r_d e^{-r_d(T-t)}\Phi(d_2) \\ &- \frac{1}{2} S_t e^{-r_f(T-t)} \frac{\sigma + 2\sigma H t^{2H-1}}{\sqrt{T - t + T^{2H} - t^{2H}}}\Phi'(d_1), \\ \Gamma &= \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)} \frac{\Phi'(d_1)}{S_t\sqrt{\sigma^2(T - t + T^{2H} - t^{2H})}}, \\ \vartheta_\sigma &= \frac{\partial C}{\partial \sigma} = S_t e^{-r_f(T-t)}\Phi'(d_1)\sqrt{T - t + T^{2H} - t^{2H}}. \end{aligned}$$

The following theorem represents the influence of Hurst parameter $H$ in $MFBM$ model.

**Theorem 3.11.** The impact of the parameter $H$ is given by

$$\frac{\partial C}{\partial H} = S_t e^{-r_f(T-t)} \frac{\sigma(T^{2H}\ln T - t^{2H}\ln t)}{\sqrt{T - t + T^{2H} - t^{2H}}}\Phi'(d_1). \tag{32}$$

## 3.5 The $JMFBM$ model

The $JMFBM$ is a mixed of Poisson jumps and the $MFBM$ for capture behaviour of financial market such as:fluctuations,long-range correlations,jumps and etc. This model is introduced by Shokrollahi and Kılıçman (2014a). The stock price of this model as follows

$$\begin{aligned} dS_t &= (r_d - r_f)S_t dt + \sigma dB(t) + \sigma S_t dB_H(t) + (e^{J(t)} - 1)dN_t, \\ & S_0 = S > 0, \quad 0 \leq t \leq T, \end{aligned} \tag{33}$$

where $(e^{J(t)} - 1) \sim N(\mu_{J(t)}, \delta^2(t))$. Moreover, all three source of $B_H(t)$, $N(t)$, and $e^{J(t)} - 1$, are supposed to be independent.

Then,the price of call currency option is given by

$$\begin{aligned} C(t, S_t) &= \sum_{n=0}^{\infty} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \varepsilon_n \Big[ S_t \prod_{i=1}^{n} e^{J(t_i)} e^{(-r_f + \lambda \mu_{J(t)})(T-t)} \Phi(d_1) \\ &- K e^{-r_d(T-t)} \Phi(d_2) \Big], \end{aligned} \tag{34}$$

and for the put currency option we have

$$\begin{aligned} P(t, S_t) &= \sum_{n=0}^{\infty} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \varepsilon_n \Big[ K e^{-r_d(T-t)} \Phi(-d_2) \\ &- S_t \prod_{i=1}^{n} e^{J(t_i)} e^{(-r_f + \lambda \mu_{J(t)})(T-t)} \Phi(-d_1) \Big], \end{aligned} \tag{35}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_t \prod_{i=1}^{n} e^{J(t_i)}}{K} + (r_d - r_f - \lambda \mu_{J(t)})(T-t) + \frac{\sigma^2}{2}(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma \sqrt{T - t + T^{2H} - t^{2H}}}, \\ d_2 &= d_1 - \sigma \sqrt{T - t + T^{2H} - t^{2H}}. \end{aligned} \tag{36}$$

**Theorem 3.12.** For the $JMFBM$ model, $V(t, S_t)$ is the solution of the $PDE$

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f - \lambda \mu_{J(t)}) S_t \frac{\partial V}{\partial S_t} \\ &- r_d V + \lambda E[V(e^{J(t)} S_t, t) - V(S_t, t)] = 0. \end{aligned} \tag{37}$$

**Theorem 3.13.** The Greeks for the $JMFBM$ call currency option model as follows

$$\begin{aligned}\triangle &= \frac{\partial C}{\partial S_t} = \sum_{m=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}\varepsilon_n \\ &\left[\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi(d_1)\right]. \\ \nabla &= \frac{\partial C}{\partial K} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}\varepsilon_n \left[-e^{-r_d(T-t)}\Phi(d_2)\right]. \\ \rho_{r_d} &= \frac{\partial C}{\partial r_d} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}\varepsilon_n \left[K(T-t)e^{-r_d(T-t)}\Phi(d_2)\right]. \\ \rho_{r_f} &= \frac{\partial C}{\partial r_f} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}\varepsilon_n \\ &\times \left[-S_t(T-t)\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi(d_1)\right].\end{aligned}$$

$$\begin{aligned}\Theta &= \frac{\partial C}{\partial t} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^{n+1}(T-t)^n - ne^{-\lambda(T-t)}\lambda^n(T-t)^{n-1}}{n!}\varepsilon_n \\ &\left[S_t\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2)\right] \\ &+\sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}\varepsilon_n \\ &\times\Big[(r_f+\lambda\mu_{J(t)})S_t\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi(d_1) - r_dKe^{-r_d(T-t)}\Phi(d_2) \\ &-\frac{1}{2}\frac{\sigma^2+2H\sigma^2t^{2H-1}}{\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}}S_t\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1)\Big].\end{aligned}$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}\varepsilon_n \left[\frac{\prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_J(t))(T-t)}}{S_t\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}}\Phi'(d_1)\right]$$

$$\nu_\sigma = \frac{\partial C}{\partial \sigma} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}\varepsilon_n \left[S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1)\sqrt{T-t+T^{2H}-t^{2H}}\right].$$

The following theorem shows the effect of hurst parameter $H$ in $JMFBM$.

**Theorem 3.14.** The effect of parameter $H$ is

$$\frac{\partial C}{\partial H} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}\varepsilon_n \Big[S_t \frac{\sigma^2(T^{2H}\ln T - t^{2H}\ln t)}{\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}} \times \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1)\Big]. \tag{38}$$

The following figures show that the Greeks of the $BS$ model, the $FBM$ model, and the $MFBM$ model. This numerical survey reveals that the variation discrepancy of Greeks with the expiration date and the exercise price for out-of-the-money call currency options. Suppose $S_t = 100, r_d = 0.021, r_f = 0.032, \sigma = 0.19, t = 0.1, H = 0.76, T \in [8.11, 18], K \in [101, 130]$. Figures 1-7 reflect the fact that the valuation of the Delta, Gamma, Nabla, Rho-$r_d$,Rho-$r_d$, Theta, and Vega with selected parameters, respectively . As are result, it is evident that there are various the valuations of the Greeks for the $BS$ model, $FBM$ model and $MFBM$ model.

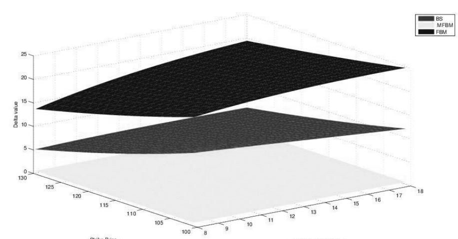


Figure 1: Delta values for the out-of-the-money case .

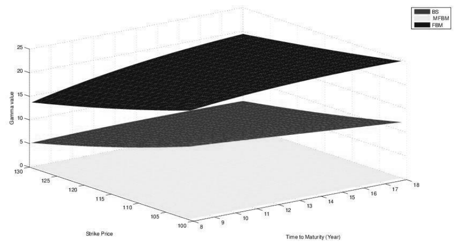


Figure 2: Gamma values for the out-of-the-money case .

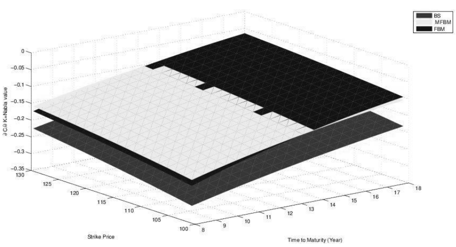


Figure 3: Nabla values for the out-of-the-money case .

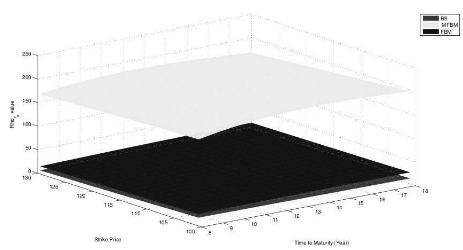


Figure 4: Rho-$r_d$ values for the out-of-the-money case .

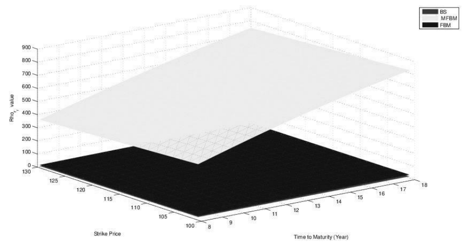


Figure 5: Rho-$r_f$ values for the out-of-the-money case .

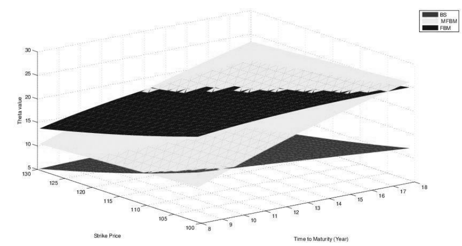


Figure 6: Theta values for the out-of-the-money case .

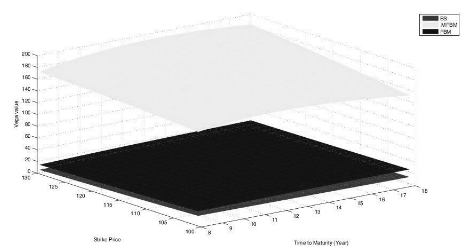


Figure 7: Vega values for the out-of-the-money case .

# 4. Conclusion

The option Greeks are accounted as substantial notions and tools in the risk management of financial portfolio. The Greeks can be employed to rebalance the portfolio to achieve desired exposure to a certain risk. More importantly, knowing the Greek, a particular exposure can be hedged from adverse changes in the market by using appropriate amount of other related financial instruments. This research presented some pricing currency option models, especially the models which are introduced in the fractional environments. The partial differential equations and the Greeks for the whole currency options models are obtained. Moreover, in the fractional environment the impacts of exponent parameter $H$ are presented.

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# Appendix

**Proof of Theorem 3.3.** Let $V(t, S_t)$ be the price of the currency derivatives at time $t$ and let $\Pi$ be the portfolio value. Thus

$$\Pi_t = V(S_t, t) - \Delta S_t. \tag{39}$$

Since

$$S_t = S_0 \exp\left[\mu T + \sigma B_T^H - \frac{1}{2}\sigma^2 T^{2H}\right]. \tag{40}$$

Then

$$\begin{aligned} D_u S_\tau &= S_\tau D_u \left(\mu\tau + \sigma B_\tau^H - \frac{1}{2}\sigma^2\tau^{2H}\right) && (41)\\ &= S_\tau [D_u(\sigma B_\tau^H)], \\ D_u^\phi &= S_\tau H \sigma \tau^{2H-1}. && (42) \end{aligned}$$

Hence we have

$$\begin{aligned} d\Pi_t &= dV(t, S_t) - \Delta(dS_t + r_f S_t dt) \\ &= \left(\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t \frac{\partial V}{\partial S_t}\right) dt \\ &+ \sigma S_t \frac{\partial V}{\partial S_t} dB_t^H - \Delta\left(\mu S_t dt + \sigma S_t dB_t^H + r_f S_t dt\right) \\ &= \left(\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t \frac{\partial V}{\partial S_t} - \Delta\mu S_t - \Delta r_f S_t\right) dt \\ &+ \left(\sigma S_t \frac{\partial V}{\partial S_t} - \Delta\sigma S_t\right) dB_t^H. \end{aligned} \tag{43}$$

For eliminate the stochastic noise we choose $\Delta = \frac{\partial V}{\partial S_t}$, then

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - \Delta r_f S_t\right) dt. \tag{44}$$

The return of an amount $\Pi_t$ invested in bank account equal to $r_d \Pi_t dt$ at time $dt$. for absence of arbitrage these values must be same Sun (2013), thus

$$\left(\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - r_f S_t \frac{\partial V}{\partial S_t}\right) dt = r_d \Pi_t dt. \tag{45}$$

Since $\Pi_t = V(t, S_t) - \Delta S_t$, hence

$$\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - r_f S_t \frac{\partial V}{\partial S_t} dt = r_d(V - S_t \frac{\partial V}{\partial S_t}), \tag{46}$$

so

$$\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f) S_t \frac{\partial V}{\partial S_t} - r_d V = 0. \tag{47}$$

**Proof of Theorem 3.4.** First, the popular equation is obtained. Let $y$ be one of the impression operators

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y} \Phi(d_1) + S_t e^{-r_f(T-t)} \frac{\partial \Phi(d_1)}{\partial y} \\ &- \frac{\partial K e^{-r_d(T-t)}}{\partial y} \Phi(d_2) - K e^{-r_d(T-t)} \frac{\partial \Phi(d_2)}{\partial y}. \end{aligned} \tag{48}$$

But

$$\begin{aligned} \frac{\partial \Phi(d_2)}{\partial y} &= \Phi'(d_2) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{(d_1 - \sqrt{\sigma^2(T^{2H} - t^{2H})})^2}{2} \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp\left( d_1 \sqrt{\sigma^2(T^{2H} - t^{2H})} \right) \exp\left( -\frac{\sigma^2(T^{2H} - t^{2H})}{2} \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp\left( \ln \frac{S_t}{K} + (r_d - r_f)(T - t) \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{S_t}{K} \exp\left( (r_d - r_f)(T - t) \right) \frac{\partial d_2}{\partial y}. \end{aligned} \tag{49}$$

Then we have that

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y} \Phi(d_1) - \frac{\partial K e^{-r_d(T-t)}}{\partial y} \Phi(d_2) \\ &+ S_t e^{-r_f(T-t)} \Phi'(d_1) \frac{\partial \sqrt{\sigma^2(T^{2H} - t^{2H})}}{\partial y}. \end{aligned} \tag{50}$$

Substituting in (50) we get the desired.

**Proof of Theorem 3.5.**

$$\begin{aligned}\eta &= \frac{\partial C}{\partial H} = S_t e^{-r_f(T-t)}\Phi'(d_1)\frac{\partial\sqrt{\sigma^2(T^{2H}-t^{2H})}}{\partial H} \\ &= S_t e^{-r_f(T-t)}\Phi'(d_1)\frac{\sigma(T^{2H}-t^{2H})}{\sqrt{T^{2H}-t^{2H}}}. \qquad (51)\end{aligned}$$

**Proof of Theorem 3.12.** Let $V(t, S_t)$ be the price of the currency derivatives at time $t$ and let $\Pi$ be the portfolio value. Hence

$$\Pi_t = V(S_t, t) - \Delta S_t. \qquad (52)$$

Portfolio value movements by in a very small term of time. Therefore $d\Pi_t = dV(S_t, t) - \Delta\left[dS_t + (r_f + \lambda\mu_{J(t)})S_t dt\right]$ since

$$S_t = S\exp\left[(\mu - \lambda\mu_{J(t)})t + \sigma B_t^H + \sigma B_t - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H} + \sum_{i=1}^{N_t} J(t_i)\right]. \qquad (53)$$

By applying the It formula for jump diffusion process Matsuda (2004) we have

$$\begin{aligned}dV(t, S_t) &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1}S_t^2\frac{\partial^2 V}{\partial S_t^2} \\ &+ (\mu - \lambda\mu_{J(t)})S_t\frac{\partial V}{\partial S_t} + \sigma S_t\frac{\partial V}{\partial S_t}dB_t \\ &+ \sigma S_t\frac{\partial V}{\partial S_t}dB_t^H + [V(e^{J(t)}S_t, t) - V(S_t, t)]dN_t. \qquad (54)\end{aligned}$$

The term of $[V(e^{J(t)}S_t, t) - V(S_t, t)]dN_t$ describe the discrepancy in the option price when a jump happens. The movement in the portfolio value presents in this equation

$$\begin{aligned}
d\Pi_t &= dV(t,S_t) - \Delta\left(dS_t + (r_f + \lambda\mu_{J(t)})S_t dt\right) \\
&= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (\mu - \lambda\mu_{J(t)})S_t \frac{\partial V}{\partial S_t} + \sigma S_t \frac{\partial V}{\partial S_t} dB_t \\
&+ \sigma S_t \frac{\partial V}{\partial S_t} dB_t^H + [V(e^{J(t)}S_t, t) - V(S_t, t)]dN_t \\
&- \Delta\Big[S_t(\mu - \lambda\mu_{J(t)})dt + \sigma S_t dB_t + \sigma S_t dB_t^H + S_t(e^{J(t)} - 1)dN_t - (r_f + \lambda\mu_{J(t)})S_t dt\Big] \\
d\Pi_t &= \Big[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (\mu - \lambda\mu_{J(t)})S_t \frac{\partial V}{\partial S_t} - \Delta(\mu - \lambda\mu_{J(t)})S_t \\
&- \Delta(r_f + \lambda\mu_{J(t)})S_t\Big]dt + (\sigma S_t \frac{\partial V}{\partial S_t} - \Delta\sigma S_t)dB_t + (\sigma S_t \frac{\partial V}{\partial S_t} - \Delta\sigma S_t)dB_t^H \\
&+ [V(e^{J(t)}S_t, t) - V(S_t, t) - \Delta S_t(e^{J(t)} - 1)]dN_t. \qquad (55)
\end{aligned}$$

By setting $\Delta = \frac{\partial V}{\partial S_t}$ to eliminate the stochastic noise then

$$\begin{aligned}
d\Pi_t &= \Big[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \lambda E[V(e^{J(t)}S_t, t) - V(S_t, t) \\
&- \frac{\partial V}{\partial S_t}(e^{J(t)} - 1)S_t] - \Delta(r_f + \lambda\mu_{J(t)})S_t\Big]dt. \qquad (56)
\end{aligned}$$

The return of an amount $\Pi_t$ invested in bank account equal to $r_d\Pi_t dt$ at time $dt$. For no-arbitrage these prices should be similar Sun (2013).Then

$$\begin{aligned}
&\{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \lambda E[V(e^{J(t)}S_t, t) - V(S_t, t) \\
&- \frac{\partial V}{\partial S_t}(e^{J(t)} - 1)S_t] - \Delta(r_f + \lambda\mu_{J(t)})S_t\}dt = r_d\Pi_t dt. \qquad (57)
\end{aligned}$$

Since $\Pi_t = V(t,S_t) - \Delta S_t$, thus

$$\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} \\
&+ \lambda E[V(e^{J(t)}S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t}(e^{J(t)} - 1)S_t] - (r_f + \lambda\mu_{J(t)})S_t \frac{\partial V}{\partial S_t} \\
&= r_d\left(V(t,S_t) - \frac{\partial V}{\partial S_t}S_t\right), \qquad (58)
\end{aligned}$$

so

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + H\sigma^2 t^{2H-1}S^2\frac{\partial^2 V}{\partial S^2} + (r_d - r_f - \lambda\mu_{J(t)})S\frac{\partial V}{\partial S}\\ &- r_d V + \lambda E[V(e^{J(t)}S_t, t) - V(S_t, t)] = 0. \qquad (59)\end{aligned}$$

**Proof of Theorem 3.13.** First, the popular equation is obtained. Let $y$ be one of the impression operators. By setting

$$C_1(t, S_t) = S_t\prod_{i=1}^{n} e^{J(t_i)}e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \qquad (60)$$

then we have

$$\begin{aligned}\frac{\partial C_1}{\partial y} &= \frac{\partial\left(S_t\prod_{i=1}^{n} e^{J(t_i)}e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\right)}{\partial y}\Phi(d_1)\\ &+ S_t\prod_{i=1}^{n} e^{J(t_i)}e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\frac{\partial\Phi(d_1)}{\partial y}\\ &- \frac{\partial Ke^{-r_d(T-t)}}{\partial y}\Phi(d_2) - Ke^{-r_d(T-t)}\frac{\partial\Phi(d_2)}{\partial y}. \qquad (61)\end{aligned}$$

But

$$\begin{aligned}\frac{\partial\Phi(d_2)}{\partial y} &= \Phi'(d_2)\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(d_1 - \sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})})^2}{2}\right]\frac{\partial d_2}{\partial y}.\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\exp\left[d_1\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}\right]\\ &\times \exp\left[-\frac{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}{2}\right]\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\exp\left[\ln\left(\frac{S_t\prod_{i=1}^{n} e^{J(t_i)}}{K}\right) + (r_d - r_f - \lambda\mu_{J(t)})(T-t)\right]\frac{\partial d_2}{\partial y}\\ &= \Phi'(d_1)\frac{S_t\prod_{i=1}^{n} e^{J(t_i)}}{K}\exp[(r_d - r_f - \lambda\mu_{J(t)})(T-t)]\frac{\partial d_2}{\partial y} \qquad (62)\end{aligned}$$

Then we have that

$$\begin{aligned}
\frac{\partial C_1}{\partial y} &= \frac{\partial \left(S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\right)}{\partial y}\Phi(d_1) - \frac{\partial K e^{-r_d(T-t)}}{\partial y}\Phi(d_2) \\
&+ \Big[S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1) \\
&\times \frac{\partial\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}}{\partial y}\Big], \qquad (63)
\end{aligned}$$

substituting in equation (63) we get the other Greeks .

**Proof of Theorem 3.14.** We first differentiate $C(t, S_t)$ under to $H$ then we have

$$\begin{aligned}
\frac{\partial C}{\partial H} &= \sum_{m=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}\varepsilon_n \\
&\times \left[S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1)\frac{\partial\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}}{\partial H}\right] \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}\varepsilon_n \Big[S_t \prod_{i=1}^{n} e^{J(t_i)} e^{-(r_f+\lambda\mu_{J(t)})(T-t)}\Phi'(d_1) \\
&\times \frac{\sigma^2(T^{2H}\ln T - t^{2H}\ln t)}{\sqrt{\sigma^2(T-t)+\sigma^2(T^{2H}-t^{2H})}}\Big]. \qquad (64)
\end{aligned}$$

Proof of the other Theorems are the same with the mentioned above proofs.